Sloshing of Liquids in Cylindrical Tanks

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Introduction

THE sloshing frequencies of liquids in cylindrical tanks are important design parameters for aircraft and space vehicles. For instance, launch vehicles have an enormous percentage of their initial weight as fuel; consequently, the dynamic forces resulting from the motions of these large liquid masses can be substantial even to the point of exceeding the ability of the control system to counteract them or of the structure to resist them.[1] In order to obtain good estimates of the magnitude of the forces generated by possible couplings between the control system, the structural frame, and the sloshing fuel, it is desirable to know the natural frequencies of the system as accurately as possible. In this Note, we discuss a method recently developed by the authors[2] that leads to a simple but effective procedure for estimating the sloshing frequencies with rigorous error bounds.

An important previous contribution to this problem was made by Budiansky,[3] who used conformal mapping to formulate the problem as an integral equation which he then discretized. Out calculations improve his results.

Sloshing Problem

The governing equations are[4]

$$\Delta u = 0 \text{ in } R, \quad \frac{\partial u}{\partial u} = 0 \text{ on } S, \quad \frac{\partial u}{\partial n} = \sigma u \text{ on } T \tag{1}$$

where u is the velocity potential of an incompressible, inviscid fluid in a tank R with rigid walls S and free surface T. Here Δ is the Laplace operator, $\partial/\partial n$ is the outer normal derivative, and $\sigma = \omega^2/g$ gives the frequencies ω of oscillation with g the acceleration of gravity.

We consider a long cylindrical tank of circular cross section as shown in Fig. 1. Gravity acts downward which we have taken as the x direction. The circle has unit radius and is centered at the origin. The tank is partially filled and has the free surface at $x = h$.

The Method of A Posteriori/A Priori Inequalities

Problem (1) has characteristic values

$$0 = \sigma_0 < \sigma_1 < \sigma_2 < \ldots$$

with associated characteristic modes $u_0 = \text{const}$, $u_1, u_2, \ldots$, which may be assumed normalized so that

$$\int_T u_i u_j \, dy = \delta_{ij}$$

Any continuous function u_* defined on T can be expanded in a series in the u_i and

$$\int_T u_*^2 \, dy = \sum_{i=0}^{\infty} \left(\int_T u_* u_i \, dy \right)^2 \tag{2}$$

It follows from Eq. (2) that if

$$\int_T u_* \, dy = 0$$

then, for any number σ_*,

$$\min_{i \neq 0} \left(\frac{\sigma_i - \sigma_*}{\sigma_i} \right)^2 \int_T u_*^2 \, dy \le \sum_{i=1}^{\infty} \left(\frac{\sigma_i - \sigma_*}{\sigma_i} \int_T u_* u_i \, dy \right)^2 \tag{3}$$

Let the function w be defined by

$$\Delta w = \Delta u_* \text{ in } R, \quad \frac{\partial w}{\partial n} = \frac{\partial u_*}{\partial n} \text{ on } S,$$

$$\frac{\partial w}{\partial n} = \frac{\partial u_*}{\partial n} - \sigma_* u_* \text{ on } T, \quad \int_T w \, dy = 0 \tag{4}$$

An application of Green's identity shows the right-hand term in Eq. (3) is equal to

$$\int_T w^2 \, dy$$

Thus, we have the a posteriori estimate for an approximate characteristic value σ_* and approximate mode u_*:

$$\min_{i \neq 0} \left(\frac{\sigma_i - \sigma_*}{\sigma_i} \right)^2 \le \int_T w^2 \, dy \Big/ \int_T u_*^2 \, dy \tag{5}$$

where w satisfies Eq. (4). Rather than compute w, we use an appropriate a priori inequality. If $\Delta w = \Delta u_* = 0$ in R, it can be shown[5] that

$$\int_T w^2 \, dy \le \sigma_1^{-1} (1 + \sigma_1^{-1}) \int_{SuT} \left(\frac{\partial w}{\partial n} \right)^2 ds \tag{6}$$

Combining Eqs. (3-6) gives the a posteriori/a priori inequality

$$\min_{i \neq 0} \left(\frac{\sigma_i - \sigma_*}{\sigma_i} \right)^2 \le \sigma_1^{-1} (1 + \sigma_1^{-1})$$

$$\times \left[\int_T \left(\frac{\partial u_*}{\partial n} - \sigma_* u_* \right)^2 dy + \int_S \left(\frac{\partial u_*}{\partial n} \right)^2 ds \right] \Big/ \int_T u_*^2 \, dy \tag{7}$$

for any u_* satisfying

$$\Delta u_* = 0 \text{ in } R \text{ and } \int_T u_* \, dy = 0$$

Now the right-hand side of Eq. (7) is a ratio of quadratic forms in u_*. Thus, we may take u_* to be a linear combination

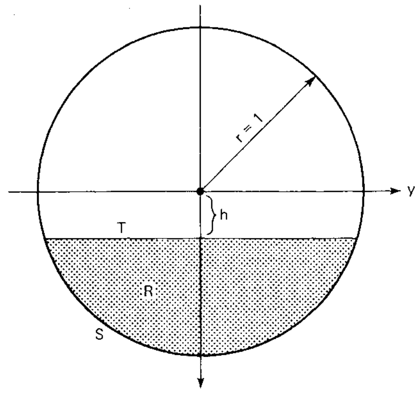

Fig. 1 Partially filled cylindrical tank.

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of convenient trial functions

$$u_* = \sum_{i=1}^{N} a_i \phi_i$$

and minimize the right side of Eq. (7) with respect to the coefficients a_i as in the Rayleigh-Ritz method, Then,

$$\min_{i \neq 0} \left| \frac{\sigma_i - \sigma_*}{\sigma_i} \right| \leq \epsilon \tag{8}$$

where

$$\epsilon = \sqrt{\sigma_1^{-1}(1+\sigma_1^{-1})\lambda}$$

and λ is the smallest eigenvalue of the relative matrix eigenvalue problem

$$Ma = \lambda Na \tag{9}$$

with

$$M_{ij} = \int_T \left(\frac{\partial \phi_i}{\partial n} - \sigma_* \phi_i\right)\left(\frac{\partial \phi_j}{\partial n} - \sigma_* \phi_j\right) \mathrm{d}y + \int_S \frac{\partial \phi_i}{\partial n}\frac{\partial \phi_j}{\partial n} \mathrm{d}s$$

$$N_{ij} = \int_T \phi_i \phi_j \mathrm{d}y$$

$$a = (a_1, a_2, \ldots, a_N)^T$$

When $\epsilon < 1$, Eq. (8) says that there is an eigenvalue σ_i satisfying

$$\sigma_*/(1+\epsilon) \leq \sigma_i \leq \sigma_*/(1-\epsilon) \tag{10}$$

thus giving a lower and an upper bound on the eigenvalue.

Note that σ_1 appearing on the right side of Eq. (7) is one of the very quantities we are approximating. For purposes of the inequality, a lower bound for σ_1 is used. Initially, a crude bound from Ref. 6 can be used until an improved bound is found by the method itself.

Numerical Results

The trial functions used were harmonic polynomials. Because of the symmetry of the cylindrical tank, the sloshing modes are either odd or even functions of y. Thus, the odd modes were approximated by taking ϕ_k to be imaginary part of $(x+iy)^k$, i.e., y, $2xy$, $3x^2y - y^3, \ldots$, and the even modes were approximated by taking ϕ_k to be the real part of $(x+iy)^k$, i.e., 1, x, $x^2-y^2, \ldots$. The integrals required to calculate the matrices M and N are then elementary, and are conveniently calculated using the recursion formulas

$$\int_T \mathrm{Re}(z^k)\,\mathrm{Re}(z^\ell)\mathrm{d}y + \int_T \mathrm{Im}(z^k)\,\mathrm{Im}(z^\ell)\mathrm{d}y$$

$$= \frac{\ell}{k+1}\left[\int_T \mathrm{Re}(z^{k+1})\,\mathrm{Re}(z^{\ell-1})\,\mathrm{d}y\right.$$

$$\left. + \int_T \mathrm{Im}(z^{k+1})\,\mathrm{Im}(z^{\ell-1})\,\mathrm{d}y\right] + \frac{2}{k+1}\sin(k-\ell+1)\theta$$

$$\int_T \mathrm{Re}(z^k)\,\mathrm{Re}(z^\ell)\mathrm{d}y - \int_T \mathrm{Im}(z^k)\,\mathrm{Im}(z^\ell)\mathrm{d}y$$

$$= \frac{2}{k+\ell+1}\sin(k+\ell+1)\theta \tag{11}$$

where $\theta = \cos^{-1} h$.

Table 1 Estimates for characteristic values of σ_i, lower and upper bounds given by $\sigma_i/(1 \pm \epsilon)$ Budiansky's values given for comparison

h	i	σ_i	ϵ	Budiansky
0.6	1	1.0970	1.4×10^{-5}	1.099
	2	2.8905	8.3×10^{-5}	
	3	4.937	7.1×10^{-4}	4.97
	4	6.990	4.5×10^{-3}	
	5	9.007	1.5×10^{-2}	9.13
0.4	1	1.1627	5.6×10^{-5}	1.165
	2	2.8892	2.6×10^{-4}	
	3	4.699	2.4×10^{-4}	4.74
	4	6.460	2.3×10^{-3}	
	5	8.199	3.7×10^{-3}	8.33
0.2	1	1.2460	2.5×10^{-4}	1.249
	2	2.9325	7.6×10^{-4}	
	3	4.607	1.2×10^{-3}	4.65
	4	6.236	9.7×10^{-4}	
	5	7.854	2.5×10^{-3}	7.99
0.0	1	1.3557	1.1×10^{-3}	1.360
	2	3.0331	1.5×10^{-3}	
	3	4.651	3.7×10^{-3}	4.70
	4	6.239	3.8×10^{-3}	
	5	7.820	5.8×10^{-3}	7.96
−0.2	1	1.5075	1.6×10^{-3}	1.513
	2	3.2164	3.8×10^{-3}	
	3	4.851	6.1×10^{-3}	4.91
	4	6.467	1.3×10^{-2}	
	5	8.078	1.8×10^{-2}	8.23
−0.4	1	1.7346	5.3×10^{-3}	1.742
	2	3.5375	9.7×10^{-3}	
	3	5.277	2.0×10^{-2}	5.34
	4	7.000	2.5×10^{-2}	
	5	8.722	5.8×10^{-2}	8.89
−0.6	1	2.1237	8.7×10^{-3}	2.13

Approximations for the first five nonzero characteristic values σ_i are given in Table 1 for the circular tank of unit radius filled to various depths. (For a tank of radius R, divide these numbers by R.) The parameter h is the distance of the free surface below the centerline, so $h>0$ is less than half-full, $h<0$ is more than half-full. Also given are the values of ϵ giving the lower and upper error bounds $\sigma_i/(1\pm\epsilon)$. Typically, 20-50 trial functions were used.

Odd-numbered σ_i correspond to odd modes and even-numbered σ_i to even modes. We also give Budiansky's[3] estimates for comparison (he computed only odd σ_i). Wee see that these numbers are slightly high, usually lying above the upper bound $\sigma_i/(1-\epsilon)$. We also mention that for the special case of the half-full cylinder, very accurate values of σ_i have been computed by another method.[7] These are (exact to the places shown):

$$\sigma_1 = 1.35573, \quad \sigma_2 = 3.03310, \quad \sigma_3 = 4.65105,$$

$$\sigma_4 = 6.2392, \quad \sigma_5 = 7.8198$$

and our numbers agree to the places shown in Table 1.

The optimum value of σ_* in Eq. (7) was obtained by alternately minimizing Eq. (7) with respect to the coefficients a_i and σ_*, i.e., for fixed σ_* the a_i are determined from Eq. (9) and, then, for the u_* held fixed, σ_* is produced from

$$\sigma_* = \int_T u_* \frac{\partial u_*}{\partial n}\mathrm{d}y \bigg/ \int_T u_*^2 \mathrm{d}y \tag{12}$$

The eigenvalue λ in Eq. (9) was computed by iteratively solving

$$Ma_{k+1} = Na_k \tag{13}$$

for the vector a, with the eigenvalue given by the Rayleigh quotient

$$\lambda = a^T Ma / a^T Na$$

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Comparison of Temperature and Velocity Spectra in a Slightly Heated Turbulent Plane Jet

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Introduction

In their experimental investigation of spectra and cospectra of velocity and temperature fluctuations in a slightly heated turbulent boundary layer, Fulachier and Dumas[1] and Fulachier[2] found that, except in the vicinity of the wall, the spectrum of temperature differed significantly from the spectrum of the longitudinal velocity component, the temperature spectrum being shifted to higher frequencies. A close analogy was found, however, between the spectral distribution of the temperature variance $\overline{\theta^2}$ and the spectral distribution of $\overline{q^2}$ or twice the turbulent kinetic energy ($\overline{q^2} \equiv \overline{u^2} + \overline{v^2} + \overline{w^2}$, u is in the x or longitudinal direction, v is in the y or mean shear direction, w is in the spanwise direction). This analogy is equivalent to one between the autocorrelation of θ and that of the fluctuating velocity vector. Fulachier[3] found that the analogy also worked in the case of turbulent boundary layer downstream of a step change in surface temperature or surface suction.

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The search for an analogy between $\overline{q^2}$ and $\overline{\theta^2}$, both of which are scalar quantities, is more attractive if not more relevant than the analogy between one component, usually $\overline{u^2}$, of the Reynolds stress tensor and $\overline{\theta^2}$. From a point of view of transport equations, Corrsin[4] had earlier noted the analogy that existed between the equations for $\overline{q^2}$ and $\overline{\theta^2}$, apart from the appearance of the pressure term in the $\overline{q^2}$ equation. Equations for two-point correlations of temperature and of the fluctuating velocity vector[2] more closely reflect this analogy, at least in the case of homogeneous turbulence when the term containing the pressure fluctuations disappears.

Fulachier and Dumas[1] concentrated especially on the low-frequency range that accounts for the major part of $\overline{q^2}$ and $\overline{\theta^2}$ in a boundary layer. In this Note, we seek to establish the validity of the analogy in the self-preserving region of a free shear flow: a turbulent plane jet. It is pertinent to point out that the boundary conditions for the velocity and thermal fields differ in an important way between a flow over a heated wall and a heated free shear flow. The presence of the wall in the boundary layer ensures a close link, near the wall, between spectra of u and θ fluctuations at every streamwise location in the flow. Such a link is absent in a jet.

Experimental Arrangement and Conditions

Velocity fluctuations u and v and the temperature fluctuation θ were measured with an X-probe/cold-wire arrangement at a distance $x = 40d$ ($d = 12.7$ mm is the nozzle width; the nozzle height is 250 mm) from the nozzle exit. The hot wires (5 μm Pt-10% Rh, 0.6-mm long) were separated by about 0.5 mm. The 0.63-μm Pt-10% Rh cold wire (0.6-mm long) was positioned about 0.5 mm upstream of the center of the X probe, orthogonally to the X probe. For the measurements of w, the cold wire was removed and the X probe, rotated so that the wires were in the x-z plane, was used in the unheated jet. The hot wires were operated with constant temperature anemometers at an overheat ratio of 1.8 while the cold wire was operated with a constant current (0.1 mA) circuit. Details of the apparatus and experimental procedure are given in Ref. 5. Hot- and cold-wire voltages were digitized for a duration of approximately 60 s, at a sampling frequency of 333 Hz into a PDP 11/34 computer. The hot-wire voltages, for the measurement of u and v fluctuations, were linearized on the computer following the removal of the temperature contamination. The velocity sensitivity of the cold wire was negligible.

The nominal jet speed U_j was 9 ms^{-1} and the nominal jet temperature, relative to ambient, was 25 K. At $x/d = 40$, the mean velocity U_0 at the jet centerline was about 3.4 m/s and the velocity half-width L_u was 0.6 m. Self-preservation of the jet was established at $x/d \simeq 20$ from mean velocity and mean temperature profiles, distributions of Reynolds stresses and heat fluxes.

Experimental Results and Discussion

Spectra computed using a fast Fourier transform algorithm are presented for two values of the ratio $\eta \equiv y/L_u$ (0 and 0.5), y being measured from the jet centerline. The value of 0.5 for η corresponds to a location where the kinematic Reynolds shear stress $\overline{uv}$ and the thermometric heat flux $\overline{v\theta}$ are relatively large, although not quite maximum. The flow is fully turbulent (the intermittency factor is unity) for values of η extending up to about 1.

The spectral density F_α is defined such that

$$\int_0^\infty F_\alpha(\omega^*)\mathrm{d}\omega^* = I$$

with $\omega^* = \omega L_u / U_0$ where ω is the circular frequency. The subscript α stands for u, v, w, θ, or q. For convenience, we